BOUSSINESQ TYPE PROBLEMS FOR THE ANISOTROPIC HALF-SPACE

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1. Starting relations. Let us consider the equilibrium equations of an anisotropic medium in the absence of body forces, singling out the variable z and omitting the summation sign

$$a_{\tau k}^{pq}\frac{\partial^2 u_\tau}{\partial x^p \partial x^q} + a_{\tau k}^{p3}\frac{\partial^2 u_\tau}{\partial x^p \partial z} + a_{\tau k}^{33}\frac{\partial^2 u_\tau}{\partial z^2} = 0 \qquad \begin{pmatrix} k, \tau = 1, 2, 3 \\ pq = 1, 2 \end{pmatrix} \tag{1.1}$$

Here $a_{\tau k}^{sr}$ are the elastic constants and u_τ are the elastic displacements. We will construct the solution to the system of equations (1.1) in the form

$$u_\tau(x^1, x^2, z) = \int_0^{2\pi} u_\tau^*(\xi, z, \lambda)\, d\lambda, \qquad \xi = x^1\cos\lambda + x^2\sin\lambda \tag{1.2}$$

Essentially this exploits the principle of superposition, formulated for the wave equation by Sobolev [1]. We will call u_τ^* the transform of u_τ. Using the standard symbol for correspondence, we get

$$\frac{\partial^2 u_\tau}{\partial x^p \partial x^q} \doteqdot \alpha_p\alpha_q\frac{\partial^2 u_\tau^*}{\partial \xi^2}, \quad \frac{\partial^2 u_\tau}{\partial x^p \partial z} \doteqdot \alpha_p\frac{\partial^2 u_\tau^*}{\partial \xi \partial z}, \quad \frac{\partial^2 u_\tau}{\partial z^2} \doteqdot \frac{\partial^2 u_\tau^*}{\partial z^2} \qquad \begin{pmatrix} \alpha_1 = \cos\lambda \\ \alpha_2 = \sin\lambda \end{pmatrix} \tag{1.3}$$

Equations (1.1) are satisfied, if

$$a_{\tau k}^{pq}\alpha_p\alpha_q\frac{\partial^2 u_\tau^*}{\partial \xi^2} + a_{\tau k}^{p3}\alpha_p\frac{\partial^2 u_\tau^*}{\partial \xi \partial z} + a_{\tau k}^{33}\frac{\partial^2 u_\tau^*}{\partial z^2} = 0 \tag{1.4}$$

We will construct the solution to this system of equations in the form

$$u_\tau^*(\xi, z, \lambda) = f_\tau(\Omega, \lambda), \qquad \Omega = \xi + \nu z \tag{1.5}$$

We get

$$(a_{\tau k}^{pq}\alpha_p\alpha_q + a_{\tau k}^{p3}\alpha_p\nu + a_{\tau k}^{33}\nu^2) f_\tau'' = 0, \qquad f_\tau'' = \frac{\partial^2 f_\tau}{\partial \Omega^2} \tag{1.6}$$

Hence

$$\Delta(\nu) = \| a_{\tau k}^{33}\nu^2 + a_{\tau k}^{p3}\alpha_p\nu + a_{\tau k}^{pq}\alpha_p\alpha_q \| = 0 \tag{1.7}$$

For example, for orthotropic bodies

$$\Delta(\nu) = \begin{vmatrix} A\alpha_1^2 + N\alpha_2^2 + M\nu^2 & (N+H)\alpha_1\alpha_2 & (M+G)\alpha_1\nu \\ (N+H)\alpha_1\alpha_2 & N\alpha_1^2 + B\alpha_2^2 + L\nu^2 & (L+F)\alpha_2\nu \\ (M+G)\alpha_1\nu & (L+F)\alpha_2\nu & M\alpha_1^2 + L\alpha_2^2 + C\nu^2 \end{vmatrix} \qquad (1.8)$$

Here $A, B, C, L, M, N, F, G, H$ — are elastic constants [2].

We assume the roots of Equation (1.7), occuring in complex conjugate pairs, to be distinct with nonvanishing imaginary parts for all α_p, which requires certain restrictions upon the elastic constants. We will not stop here to clarify this question. With this the real solution to Equations (1.4) will be written in the form

$$u_\tau^* (\xi, z, \lambda) = \operatorname{Re} \Delta_\tau^\rho (\alpha_p)\, \omega_\rho (\xi + \nu_\rho z, \lambda) \qquad (1.9)$$

Here Δ_τ^ρ are the minors of the determinant (1.7) corresponding to the elements of any row, and a $\omega_\rho(\Omega_\rho, \lambda)$ are arbitrary functions, analytic in the upper half-plane if the anisotropic medium occupies the half-space $z \geqslant 0$.

2. The fundamental problems. In the first fundamental problem the stresses

$$\sigma_z = \Phi_3 (x^1, x^2), \qquad \tau_{zx^k} = \Phi_k (x^1, x^2) \qquad (k = 1, 2) \qquad (2.1)$$

where Φ_l is a known function, are given on the boundary $z = 0$ of the half-space. With the help of Hooke's law the connection between u_τ^* and the transforms of the components of the stress tensor is easily established.

For example, for orthotropic bodies, on the boundary $z = 0$ we have

$$G \frac{\partial u_1^*}{\partial \xi} + F\alpha_2 \frac{\partial u_2^*}{\partial \xi} + C \frac{\partial u_3^*}{\partial z} = \Phi_3^*, \qquad \frac{\partial u_k^*}{\partial z} + \alpha_k \frac{\partial u_3^*}{\partial \xi} = \Phi_k^* \quad (k = 1, 2) \quad (2.2)$$

Here Φ_l^* is the transform of the function Φ_l. We are led to the following problem: to find functions ξz, analytic in the upper half of the ξz plane, vanishing at infinity, and satisfying the given combinations (2.2) on the boundary $z = 0$. It is solved very simply if Φ_l^*. is known. Substituting u_τ^* in (2.2) we get

$$\operatorname{Re} D_j^\rho (\alpha_p)\, \omega_\rho' (\xi, \lambda) = \Phi_j^* (\xi, \lambda), \qquad \omega_\rho' = \partial\omega_\rho / \partial\xi \qquad (2.3)$$

Here D_j^ρ are known quantities. Introducing functions

$$\Psi_j (\Omega, \lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi_j (t, \lambda)}{t - \Omega} dt \qquad (j = 1, 2, 3) \qquad (2.4)$$

the equality in (2.3) becomes

$$D_j^\rho (\alpha_p)\, \omega_\rho' (\xi, \lambda) = \Psi_j^+ (\xi, \lambda) \qquad (2.5)$$

where Ψ_j^+ are the limiting values of the function Ψ_j on the boundary of the half-plane. Solving (2.5) and using analytic continuation we find

$$\omega_\rho' (\Omega_\rho, \lambda) = Q_\rho^j (\alpha_p)\, \Psi_j (\Omega_\rho, \lambda), \qquad \Omega_\rho = \xi + \nu_\rho z \qquad (2.6)$$

Having found u_τ^*, with the help of the mapping formula (1.2) and by virtue of the uniqueness we have found the solution of the problem. The second fundamental problem is solved analogously.

3. Effects of axisymmetric normal loads. The above method gives the solution to the problem if a way is known for determining the transforms of the functions given on the boundary. This way is shown in the case under consideration. We have

$$\sigma_z = \Phi_3 (r), \qquad \tau_{zx^k} = 0, \qquad r^2 = (x^1)^2 + (x^2)^2 \qquad (k = 1, 2) \qquad (3.1)$$

Therefore $\Phi_k^* = 0$. For the purpose of finding Φ_3^* we point out that if the transform of any function depends only on ξ, then the function depends only on r. Indeed, we have

$$\int_0^{2\pi} \Phi^* (\xi)\, d\lambda = \int_{-\varphi}^{2\pi-\varphi} \Phi^* (r \cos \lambda)\, d\lambda \qquad \begin{pmatrix} x^1 = r\cos\varphi \\ x^2 = r \sin\varphi \end{pmatrix} \tag{3.2}$$

But

$$\frac{\partial}{\partial\varphi} \int_{-\varphi}^{2\pi-\varphi} \Phi^* (r\cos\lambda)\, d\lambda = 0 \tag{3.3}$$

Setting $\varphi = \pi$ we get

$$\int_0^{2\pi} \Phi^* (\xi)\, d\lambda = 2 \int_0^{\pi} \Phi^* (r\cos\lambda)\, d\lambda = \Phi(r) \tag{3.4}$$

Thus

$$\Phi(r) \doteqdot \Phi^*(\xi) \tag{3.5}$$

Setting $r\cos\lambda = \eta$, we write

$$\Phi(r) = 2 \int_{-r}^{r} \frac{\Phi^*(\eta)\, d\eta}{\sqrt{r^2 - \eta^2}} \tag{3.6}$$

We will assume, in addition, that $\Phi^*(-\eta) = \Phi^*(\eta)$. Then we arrive at an integral equation with respect to $\Phi^*(\eta)$, which by substitution is easily brought to the known equation of Abel. Its solution has the form

$$\Phi^*(\eta) = \frac{1}{2\pi} \frac{d}{d\eta} \int^{\eta} \frac{\Phi(r)\, r\, dr}{\sqrt{\eta^2 - r^2}} \tag{3.7}$$

Returning to the problem under consideration, we get

$$\Phi_3^*(\xi) = \frac{1}{2\pi} \frac{d}{d\xi} \int_0^{\xi} \frac{\sigma_z(r)\, r\, dr}{\sqrt{\xi^2 - r^2}}, \qquad \Phi_k^* = 0 \quad (k = 1, 2) \tag{3.8}$$

Correspondingly, we write

$$\omega_\rho' (\Omega_\rho, \lambda) = Q_\rho^3 (\alpha_p)\, \Psi_3 (\Omega_\rho) \tag{3.9}$$

The finction Ψ_3 is often found in a quite elementary way, as is seen in the examples given below.

1. Normal load, uniformly distributed over a circular area of radius R. We have

$$\sigma_z = \begin{cases} P / \pi R^2 & (0 \leqslant r \leqslant R) \\ 0 & (r > R) \end{cases} \tag{3.10}$$

where P is the pressure force. According to (3.8) we have

$$\Phi_3^*(\xi) = \begin{cases} -{}^1/_2\, P\pi^{-2}R^{-2}, & |\xi| < R \\ -{}^1/_2\, P\pi^{-2}R^{-2}\, [1 - \xi(\xi^2 - R^2)^{-1/2}], & |\xi| > R \end{cases} \tag{3.11}$$

The function Ψ_3 must vanish at infinity, and on the boundary $z = 0$ of the half-plane its real part must coincide with $\Phi_3^*(\xi)$.

All these requirements are satisfied by the function

$$\Psi_3(\Omega) = -\frac{1}{2} \frac{P}{\pi^2} \frac{d}{d\Omega} \frac{1}{\Omega + \sqrt{\Omega^2 - R^2}} \tag{3.12}$$

where by the radical is understood that branch that approaches the value Ω for large Ω. Letting R go to zero we get the $\Psi_3(\Omega)$ that corresponds to the effect on an anisotropic half-space of a normal concentrated force of intensity P. We write the complete solution for the case where the force is applied at the point $(x_0^1, x_0^2, 0)$

$$u_\tau = -\frac{P}{4\pi^2}\int_0^{2\pi} \operatorname{Re} \Delta_\tau{}^\rho Q_\rho \frac{d\lambda}{\xi - \xi_0 + \nu_p z}, \qquad \xi - \xi_0 = (x^p - x_0{}^p)\,\alpha_p \qquad (3.13)$$

2. Normal load, uniformly distributed over an annular area. If R_1 and R_2 are the radii of the annulus, then

$$\Psi_3(\Omega) = -\frac{1}{2}\frac{P}{\pi^2}\frac{d}{d\Omega}\frac{1}{\sqrt{\Omega^2 - R_1{}^2} + \sqrt{\Omega^2 - R_2{}^2}} \qquad (R_2 > R_1) \qquad (3.14)$$

From this, writing $R_1 = R_2 = R$, we get the function Ψ_3 for the case where a concentrated load uniformly distributed along a circumference of radius R acts on the half-space.

3. A normal load of given intensity, distributed over a given region on the plane $z = 0$. The solution is constructed by the method of superposition with the use of (3.13).

4. A normal load of the form

$$\sigma_z = \Phi_3\left(\sqrt{(b/a)(x^1)^2 + (a/b)(x^2)^2}\right), \qquad \tau_{zx^k} = 0 \qquad (k = 1, 2) \qquad (3.15)$$

distributed over an elliptical area with semiaxes a and b. In this case the connection between the original and its transform is established in the form

$$u_\tau = \int_0^{2\pi} u_\tau{}^*(\xi, z, \lambda)\,d\lambda, \qquad \xi = l_p\alpha_p x^p, \qquad l_1 = \frac{b}{a}, \qquad l_2 = \frac{a}{b} \qquad (3.16)$$

The solution is written in the form (1.9) and (3.9), where it is necessary to carry out everywhere the replacement of α_p by $l_p\alpha_p$. In order to make use of the result (3.8) it is necessary to introduce new variables

$$x_1{}^p = l_p x^p \ (p = 1, 2).$$

With this we immediately find the function Ψ_3 for the following three cases: a) a normal load, uniformly distributed over an elliptical area, (b) over an elliptical annulus, with boundaries of similar and similarly arranged ellipses, (c) a normal concentrated load uniformly distributed along an ellipse. In cases (a) and (c) it is sufficient to replace R by $\sqrt{ab}$ in the corresponding formulas of items 1 and 2. In case (b) it is necessary to put $R_2 = \sqrt{ab}$, $R_1 = \tau_0\sqrt{ab}$, where τ_0 is a similarity coefficient.

4. Transversely isotropic bodies. Let us dwell on this more simple case of anisotropy at greater length. We have

$$B = A, \qquad G = F, \qquad M = L, \qquad H = A - 2N \qquad (4.1)$$

The results being considered here may be obtained from the foregoing. However it is simpler to derive them directly. Let us set

$$u^* = \alpha u_0{}^* + \beta u_3{}^*, \quad v^* = \beta u_0{}^* - \alpha u_3{}^*, \quad u^* = u_1{}^*, \quad v^* = u_2{}^*, \quad \alpha = \alpha_1, \quad \beta = \alpha_2 \quad (4.2)$$

Here $u_0{}^*$, $u_3{}^*$ are new unknowns. System (1.4) is equivalent to the following:

$$A\frac{\partial^2 u_0{}^*}{\partial\xi^2} + L\frac{\partial^2 u_0{}^*}{\partial z^2} + (L+F)\frac{\partial^2 w^*}{\partial\xi\,\partial z} = 0, \qquad N\frac{\partial^2 u_3{}^*}{\partial\xi^2} + \frac{\partial^2 u_3{}^*}{\partial z^2} = 0 \qquad (4.3)$$

$$(L+F)\frac{\partial^2 u_0{}^*}{\partial\xi\,\partial z} + L\frac{\partial^2 w^*}{\partial\xi^2} + C\frac{\partial^2 w^*}{\partial z^2} = 0,$$

Conditions (2.2) take the form

$$\alpha L\left(\frac{\partial u_0^*}{\partial z}+\frac{\partial w^*}{\partial \xi}\right)+\beta L\frac{\partial u_3^*}{\partial z}=\Phi_1(\xi,\lambda),\qquad F\frac{\partial u_0^*}{\partial \xi}+C\frac{\partial w^*}{\partial z}=\Phi_3(\xi,\lambda) \quad (4.4)$$
$$\beta L\left(\frac{\partial u_0^*}{\partial z}+\frac{\partial w^*}{\partial \xi}\right)-\alpha L\frac{\partial u_3^*}{\partial z}=\Phi_2(\xi,\lambda),$$

If an axisymmetric and purely normal load acts on the boundary of the half-space, then, as is not difficult to show, $u_3^*\equiv 0$, and the problem reduces to the determination of u_0^* and w^* from the first two equations (4.3). Condition (4.4) is written in the form

$$F\frac{\partial u_0^*}{\partial \xi}+C\frac{\partial w^*}{\partial z}=\Phi_3(\xi),\qquad \frac{\partial u_0^*}{\partial z}+\frac{\partial w^*}{\partial \xi}=0 \quad (4.5)$$

The method shown above leads in the present case to the following solution of the problem:

$$u_0^*=-\operatorname{Re}\frac{L+F}{\Delta}\begin{vmatrix} A-\nu_1^2F & A-\nu_2^2F \\ \nu_1\Psi_3(\xi+\nu_1z) & \nu_2\Psi_3(\xi+\nu_2z)\end{vmatrix}$$

$$w^*=\operatorname{Re}\frac{1}{\Delta}\begin{vmatrix} A-\nu_1^2F & A-\nu_2^2F \\ (A+\nu_1^2L)\Psi_3(\xi+\nu_1z) & (A+\nu_2^2L)\Psi_3(\xi+\nu_2z)\end{vmatrix} \quad (4.6)$$

Here

$$\Delta=\begin{vmatrix} A-\nu_1^2F & A-\nu_2^2F \\ u+CL\nu_1^2 & m+CL\nu_2^2\end{vmatrix},\qquad m=CA-F(L+F) \quad (4.7)$$

The roots ν_k are found from the equation

$$\begin{vmatrix} A+\nu^2L & (L+F)\nu \\ (L+F)\nu & L+\nu^2C\end{vmatrix}=0 \quad (4.8)$$

For the effect of a concentrated force at the point $(x_0, y_0, 0)$ we get, supposing for simplicity that $\nu_k=i\gamma_k$ and $\gamma_k>0$ $(k=1,2)$

$$u(x,y,z)=-\frac{P}{4\pi^2}\frac{L+F}{\Delta_0}\begin{vmatrix} A+\gamma_1^2F & A+\gamma_2^2F \\ \gamma_1J(\xi';\alpha;\gamma_1) & \gamma_2J(\xi',\alpha;\gamma_2)\end{vmatrix}$$

$$v(x,y,z)=-\frac{P}{4\pi^2}\frac{L+F}{\Delta_0}\begin{vmatrix} A+\gamma_1^2F & A+\gamma_2^2F \\ \gamma_1J(\xi';\beta;\gamma_1) & \gamma_2J(\xi';\beta;\gamma_2)\end{vmatrix} \quad (4.9)$$

$$w(x,y,z)=\frac{P}{4\pi^2\Delta_0}\begin{vmatrix} A+\gamma_1^2F & A+\gamma_2^2F \\ (A-\gamma_1^2L)J(e';i;\gamma_1) & (A-\gamma_2^2L)J(\xi';i;\gamma_2)\end{vmatrix} \quad (4.10)$$

Here

$$\Delta_0=i\Delta,\qquad J(\xi';\delta;\gamma_k)=\operatorname{Re}\int_0^{2\pi}\frac{\delta d\lambda}{\xi'+i\gamma_kz},\qquad \xi'=(x-x_0)\alpha+(y-y_0)\beta \quad (4.11)$$

We have

$$J(\xi';\alpha;\gamma_k)=-(2\pi/\rho')T_k(\rho',z)\cos\varphi,\qquad \cos\varphi=(x-x_0)/\rho'$$
$$J(\xi';\beta;\gamma_k)=-(2\pi/\rho')T_k(\rho',z)\sin\varphi,\qquad \sin\varphi=(y-y_0)/\rho' \quad (4.12)$$
$$J(\xi';i,\gamma_k)=(2\pi/\gamma_k)z[1-T_k(\rho',z)],\qquad T_k(\rho',z)=(\rho'^2+\gamma_k^2z^2)^{-1/2}$$

Substituting (4.12) into (4.9) and (4.10), we get the solution to the problem in terms of the displacements. When we calculate the stresses we arrive at a known result, found in a different way [3]. The transition to an isotropic body is realized by setting $C=A=\lambda_0+2\mu_0$, $L+F=\lambda_0+\mu_0$ and $L=\mu_0$, where λ_0 and μ_0 are the Lame constants. In this case $\gamma_1=\gamma_2=1$. Formulas (4.9) and (4.10) lead to indeterminateness that is easily resolved. We get a known result of Boussinesq [4].

From Formula (4.10) with $z=0$ we find

$$w(x,y,0)=\frac{P\Delta_1}{2\pi\Delta_0}\frac{1}{\rho}\left(\Delta_1=\begin{vmatrix} A+F\mu_1^2 & A+F\mu_2^2 \\ A-L\mu_1^2 & A-L\mu_2^2\end{vmatrix}\right) \quad (4.13)$$

The result (4.13) permits the extension of the theory of the Hertz contact problem to the case of transversely isotropic bodies, with the condition that the bodies are compressed along the direction of their common monotropic axes.

5. Reduction to the plane problem. Using (4.2) we derive the relations for the transforms of the components of the strain tensor

$$\varepsilon_x^* = \alpha^2\varepsilon_\xi^*, \qquad \varepsilon_y^* = \beta^2\varepsilon_\xi^*, \qquad \gamma_{xy}^* = 2\alpha\beta\varepsilon_\xi^*, \qquad \varepsilon_\xi^* = \frac{\partial u_0^*}{\partial \xi}$$

$$\gamma_{zx}^* = \alpha\gamma_{z\xi}^*, \qquad \gamma_{zy}^* = \beta\gamma_{z\xi}^*, \qquad \gamma_{z\xi}^* = \frac{\partial w^*}{\partial \xi} + \frac{\partial u_0^*}{\partial z}, \qquad \varepsilon_z^* = \frac{\partial w^*}{\partial z}$$

$$\varepsilon_\xi^* = \varepsilon_x^* + \varepsilon_y^* = \alpha^2\varepsilon_x^* + \beta^2\varepsilon_y^* + \alpha\beta\gamma_{xy}^* \tag{5.1}$$

From these it is easy to extract, in transform space, the conditions of compatibility for the strain and to show that all of them will be satisfied if

$$\frac{\partial^2\varepsilon_\xi^*}{\partial z^2} + \frac{\partial^2\varepsilon_z^*}{\partial \xi^2} = \frac{\partial^2\gamma_{z\xi}^*}{\partial\xi\partial z} \tag{5.2}$$

In this same space it is possible to write down the equilibrium equations for the transforms of the components of the stress tensor, and also Hooks's law, which takes the form

$$\begin{aligned} &\sigma_x^* = (A - 2N\beta^2)\ \xi_\xi^* + F\varepsilon_z^*, \qquad \sigma_z^* = F\varepsilon_\xi^* + C\varepsilon_z^* \\ &\tau_{zx}^* = L\alpha\gamma_{z\xi}^*, \qquad \sigma_y^* = (A - 2N\alpha^2)\ \varepsilon_\xi^* + F\varepsilon_z^* \\ &\tau_{xy}^* = 2N\alpha\beta\varepsilon_\xi^*, \qquad \tau_{zy}^* = L\beta\gamma_{z\xi} \end{aligned} \tag{5.3}$$

It is easy to prove that the equilibrium equations are satisfied if

$$\frac{\partial\sigma_\xi}{\partial\xi} + \frac{\partial\tau_{z\xi}}{\partial z} = 0, \qquad \frac{\partial\tau_{z\xi}}{\partial\xi} + \frac{\partial\sigma_z}{\partial z} = 0, \qquad \sigma_\xi = \alpha^2\sigma_x^* + \beta\sigma_y^* + 2\alpha\beta\tau_{xy}^* \tag{5.4}$$

In this case

$$\sigma_\xi^* = A\varepsilon_\xi^* + F\varepsilon_z^*, \qquad \sigma_z^* = F\varepsilon_\xi^* + C\varepsilon_z^* \tag{5.5}$$

Thus, the problem of the effect on a transversely isotropic half-space of an axisymmetric normal load reduces to the plane problem (5.2), (5.4) and (5.5) with known boundary conditions.

Setting

$$\sigma_\xi^* = \partial^2 S^* / \partial z^2, \qquad \sigma_z^* = \partial S^* / \partial \xi^2, \qquad \tau_{z\xi}^* = -\,\partial^2 S^* / \partial\xi\partial z \tag{5.6}$$

with the use of (5.2) and (5.5) we derive

$$\left(\frac{\partial^2}{\partial z^2} - \nu_1^2\frac{\partial^2}{\partial\xi^2}\right)\left(\frac{\partial^2 S^*}{\partial z^2} - \nu_2^2\frac{\partial^2 S^*}{\partial\xi^2}\right) = 0 \tag{5.7}$$

Here ν_k are the roots of Equation (4.8). For an isotropic medium we get the biharmonic equation. It is not difficult to derive the corresponding equation for the original.

We have, on the basis of (1.3)

$$\sigma_z = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2}, \qquad \tau_{zx} = -\frac{\partial^2 S}{\partial z\partial x}, \qquad \tau_{zy} = -\frac{\partial^2 S}{\partial z\partial y} \tag{5.8}$$

where $S(x, y, z)$ corresponds to the transform $S^*(\xi, z, \lambda)$. In place of (5.7) we will have

$$\left[\frac{\partial^2}{\partial z^2} - \nu_1^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\left[\frac{\partial^2 S}{\partial z^2} - \nu_2^2\left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2}\right)\right] = 0 \tag{5.9}$$

Correspondingly, we obtain

$$\sigma_x = (1 - 2CD_0)\frac{\partial^2 S}{\partial z^2} + 2FD_0\frac{\partial^2 S}{\partial y^2} + 2CD_0\frac{\partial^2}{\partial z^2}\int_0^{2\pi}\alpha^2 S^*\,d\lambda$$

$$\sigma_y = (1 - 2CD_0)\frac{\partial^2 S}{\partial z^2} + 2FD_0\frac{\partial^2 S}{\partial x^2} + 2CD_0\frac{\partial^2}{\partial z^2}\int_0^{2\pi}\beta^2 S^*\,d\lambda \qquad (5.10)$$

$$\tau_{xy} = -2FD_0\frac{\partial^2 S}{\partial x\partial y} + 2CD_0\frac{\partial^2}{\partial z^2}\int_0^{2\pi}\alpha\beta S^* d\lambda, \quad D_0 = \frac{N}{AC - F^2}$$

If

$$S^* = \frac{\partial^2\Phi^*}{\partial\xi^2} \qquad (5.11)$$

which is equivalent to the replacement

$$S(x, y, z) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} \qquad (5.12)$$

then

$$\int_0^{2\pi}\alpha^2 S^*\,d\lambda = \frac{\partial^2\Phi}{\partial x^2}, \quad \int_0^{2\pi}\beta^2 S^*\,d\lambda = \frac{\partial^2\Phi}{\partial y^2}, \quad \int_0^{2\pi}\alpha\beta S^* d\lambda = \frac{\partial^2\Phi}{\partial x\partial y} \qquad (5.13)$$

and the stresses are expressed through a single function Φ, satisfying Equation (5.9).

We note that the results (5.1) to (5.6) and also (5.8) and (5.10) to (5.13) remain in effect also in the case when the elastic coefficients depend on z.

In the general case of transversely isotropic bodies the relations (4.2) correspond to the representations for the originals

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}, \qquad w = \frac{\partial\chi}{\partial z} \qquad (5.14)$$

where the function χ is introduced for convenience. The equilibrium equations will be satisfied if

$$A\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) + L\frac{\partial^2\varphi}{\partial z^2} + (L + F)\frac{\partial^2\chi}{\partial z^2} = 0$$

$$(L + F)\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) + L\left(\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2}\right) + C\frac{\partial^2\chi}{\partial z^2} = 0 \qquad (5.15)$$

$$N\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + L\frac{\partial^2\psi}{\partial z^2} = 0$$

It is easy to show that the functions φ and χ satisfy Equations (5.9).

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